D_5 and D_6

Let's see why D_n should have $2n$ elements by analysing the cases for D_5 (odd n) and D_6 (even n).

First of all, note that

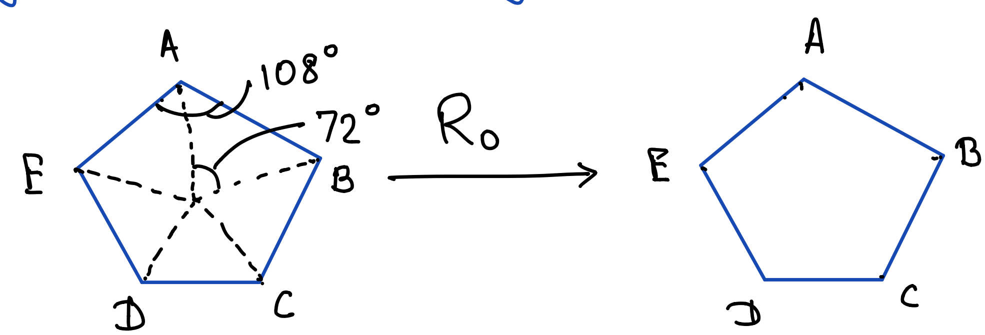
$$D_n = \{\text{symmetries of } n\text{-gon}\}$$

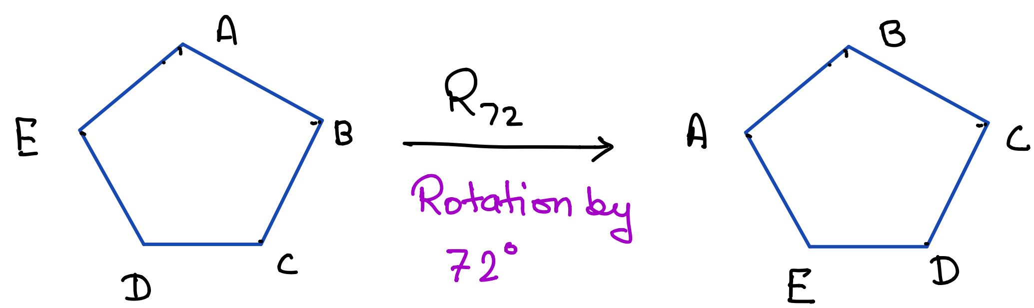
so, e.g., elements of D_4 are the symmetries of the square, and not the square itself and hence the elements are different.

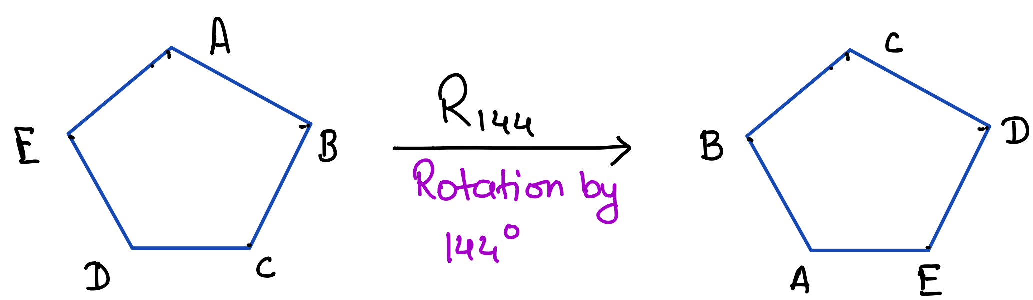
D_5 This is the group of a symmetries of a regular pentagon and we want to see that why should it have 10 elements.

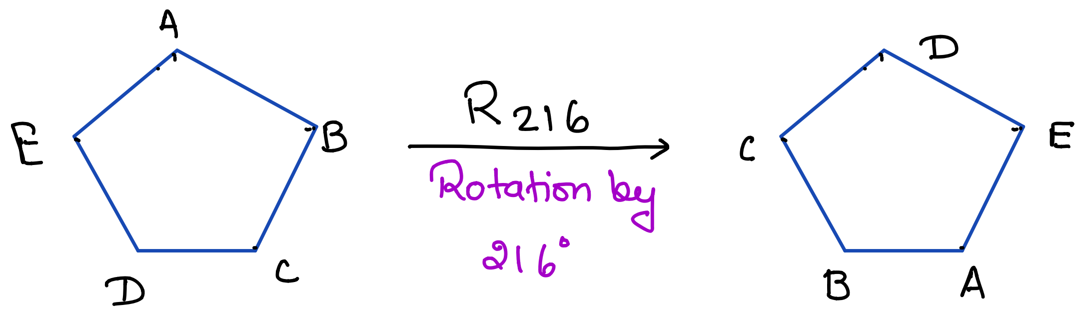
Note that the angle made at the centre of a regular pentagon is $72°$.

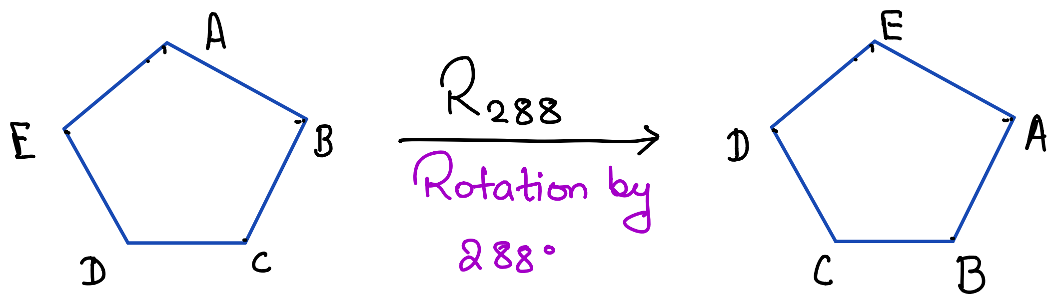
So, we can rotate the pentagon 5 times, as $\frac{360^\circ}{72^\circ} = 5$, so after rotating 5 times, we'll get back to identity.





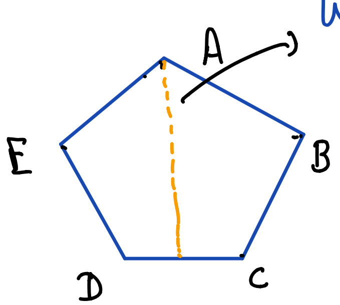






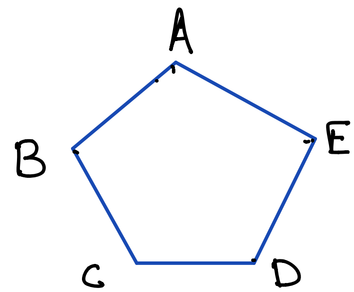
One more rotation by 72° will give us the same result as R_0. So we got 5 elements of D_5, R_0, R_{72}, R_{144}, R_{216} and R_{288}.

Let's see how to get the remaining elements by flipping.

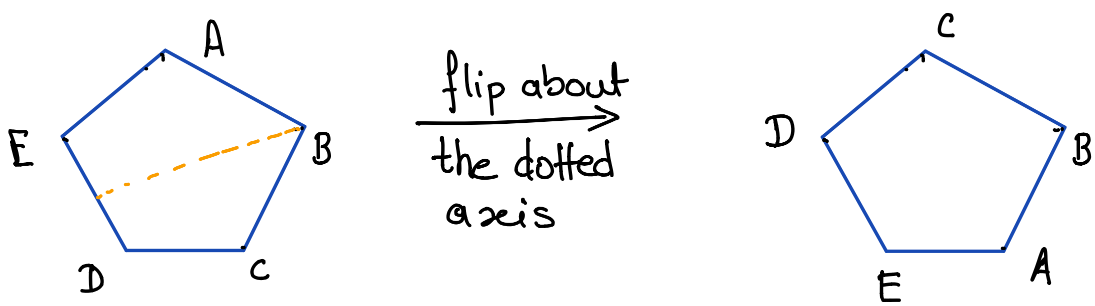


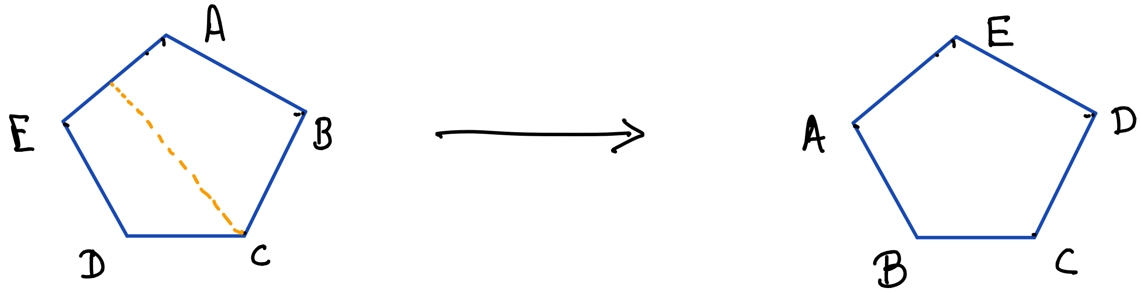
We can flip about this axis which starts from A and ends at the midpoint of the edge opposite to A which is DC.

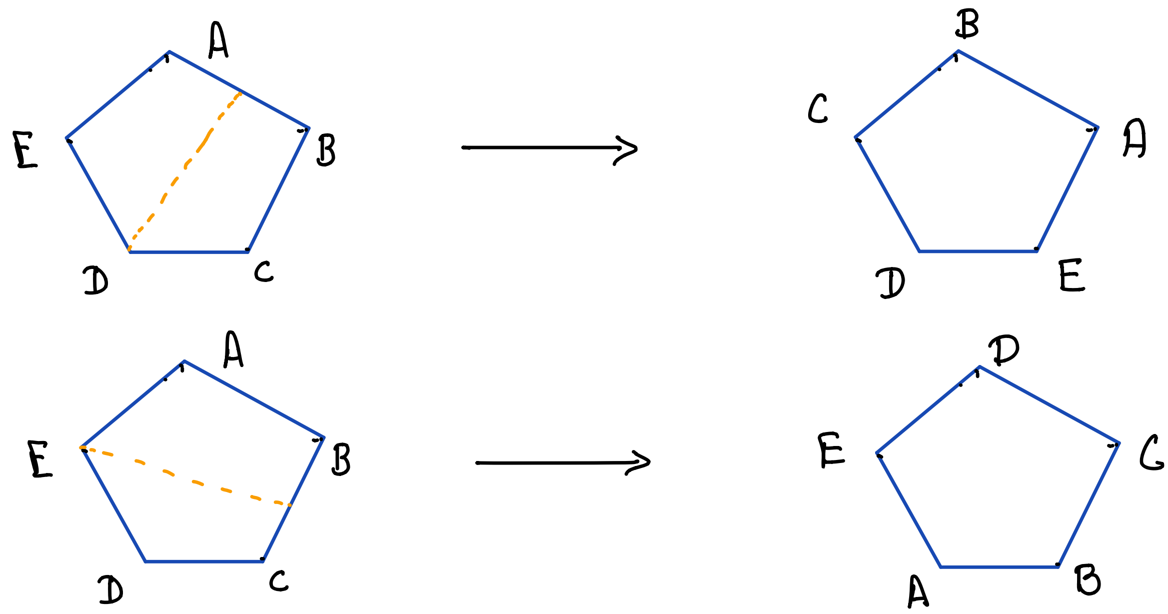
The result will be



i.e., apart from A, all vertices got flipped.
But, there are total 5 vertices, A, B, C, D, E.
So we can do the same thing with any of them, i.e, flip about an axis which starts from one of the vertex and ends at the midpoint of the edge of the opposite edge. So we get





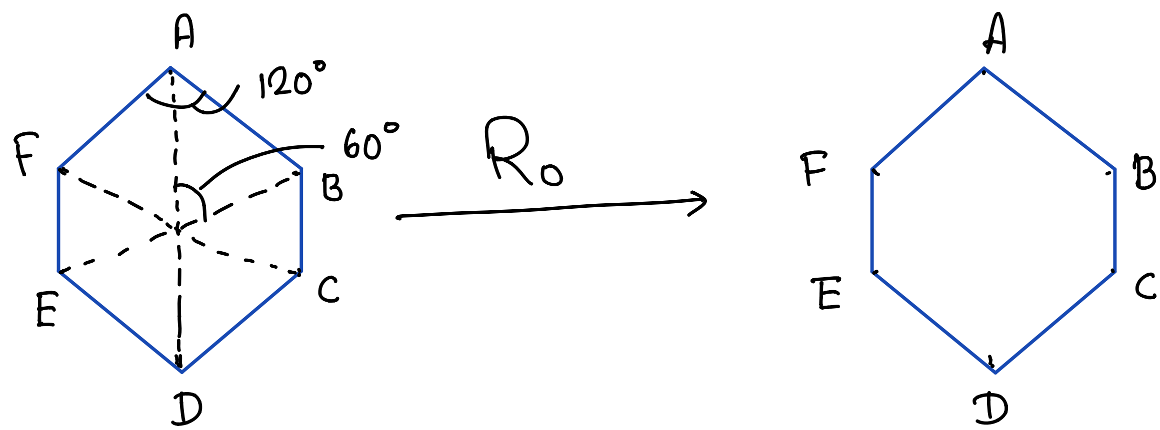


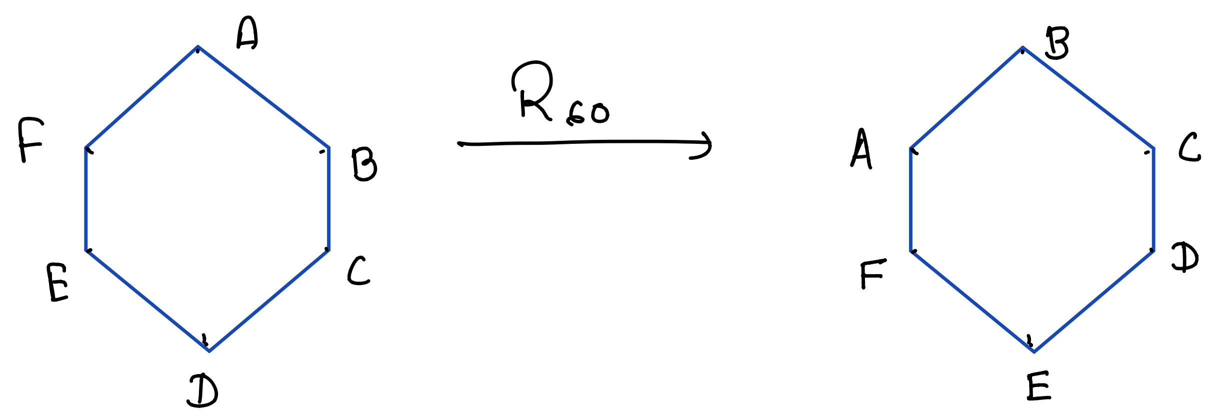
So we get the remaining 5 elements and hence a total of 10 elements.

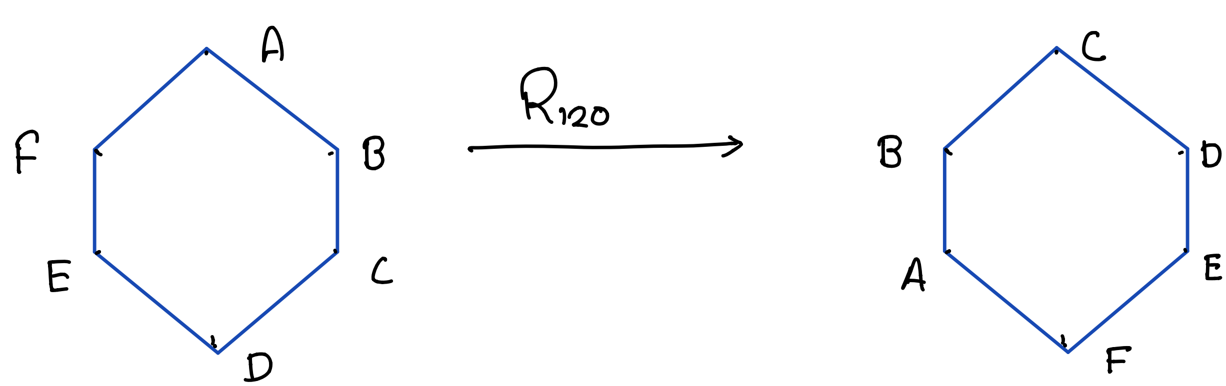
The general situation of D_n, n odd is similar.

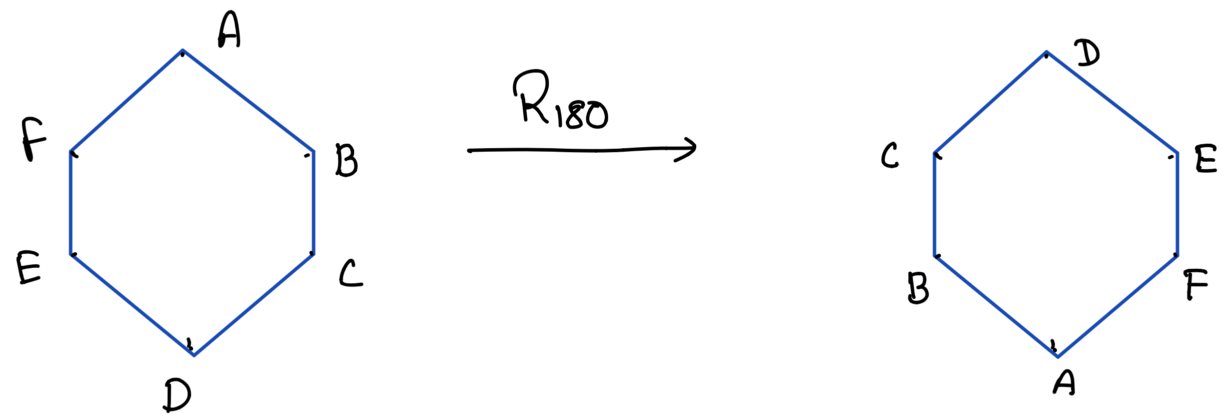
D_6 Now let's see D_6, which is the group of symmetries of a regular hexagon. It has 12 elements. Let's see why. The angle made at the centre of a regular hexagon is $60°$.

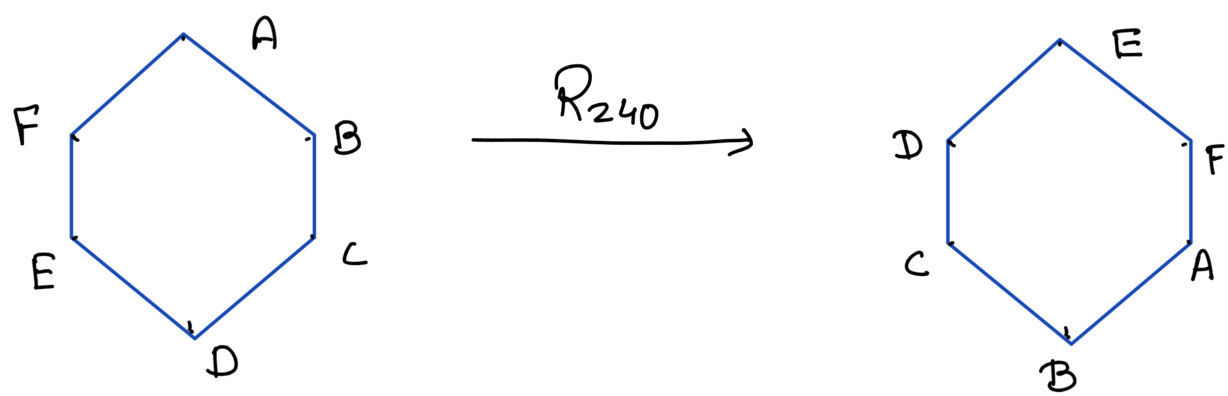
So, we can rotate $\frac{360^\circ}{60^\circ} = 6$ times, which will give the 6 elements R_0, R_{60}, R_{120}, R_{180}, R_{240} and R_{300}.

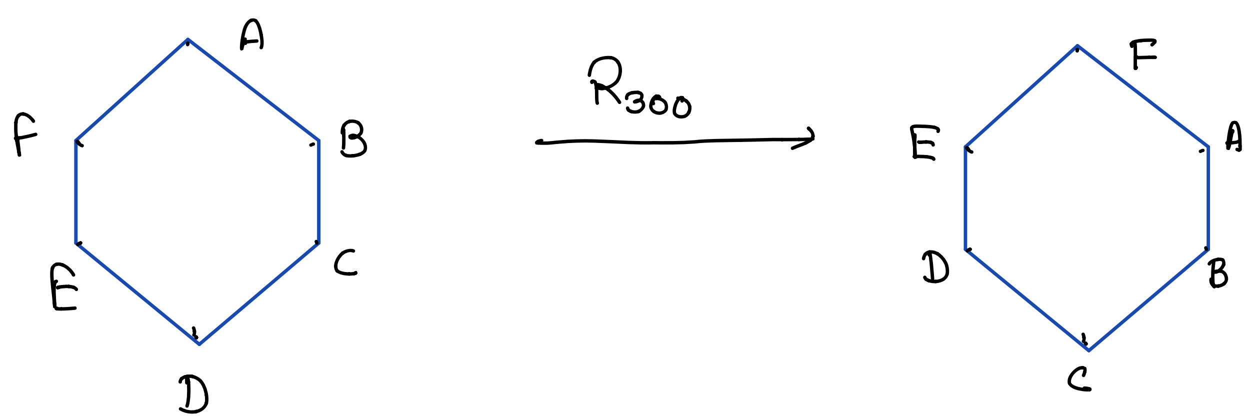






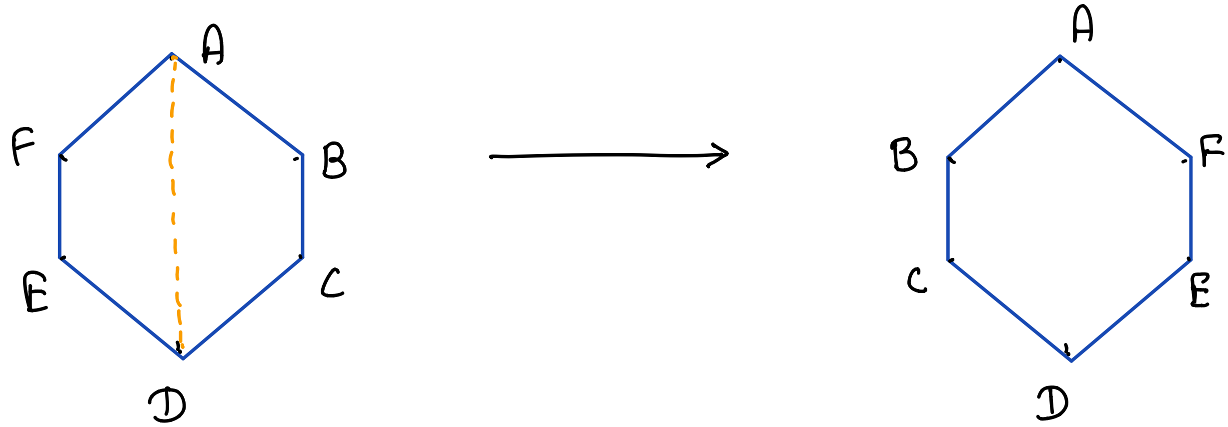




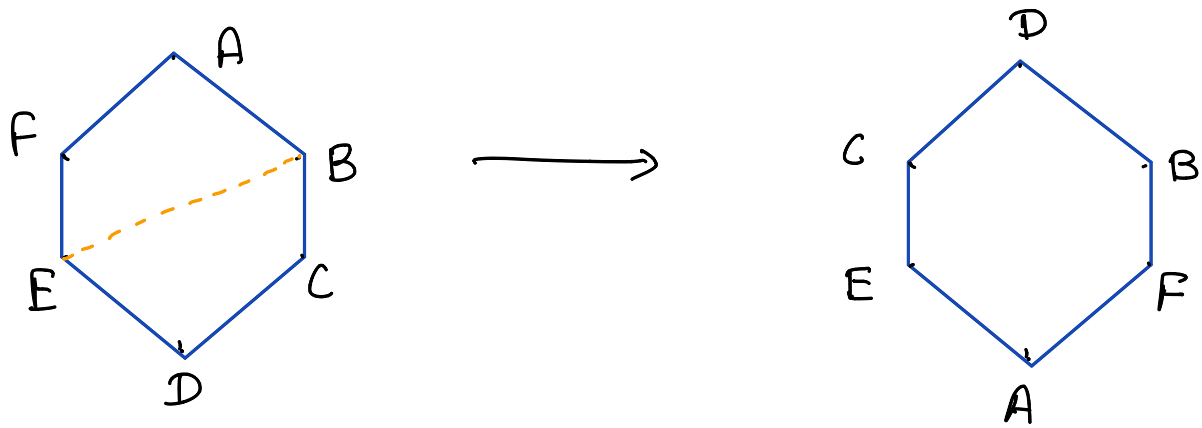


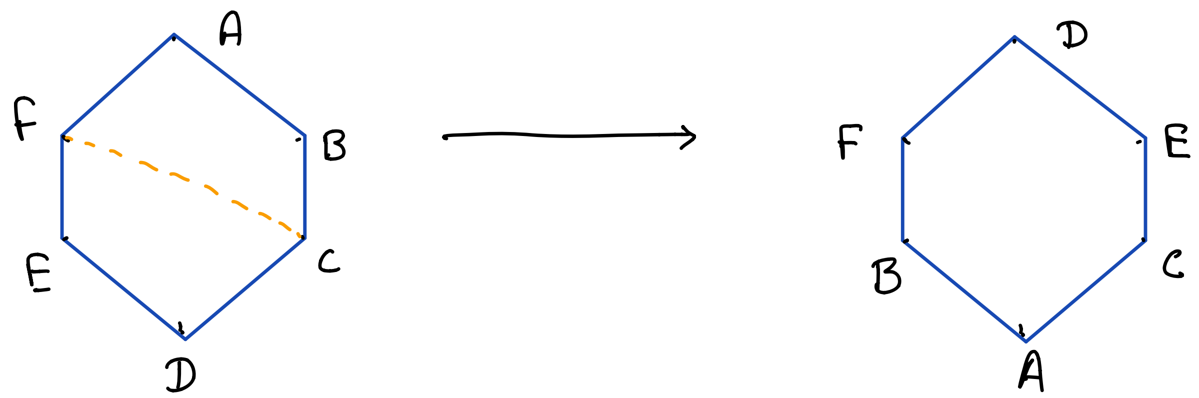
One more rotation of 60° will give the same result as R_0. Now, we'll see the flippings. We can draw an axis joining the vertices A and

D and flip about that.

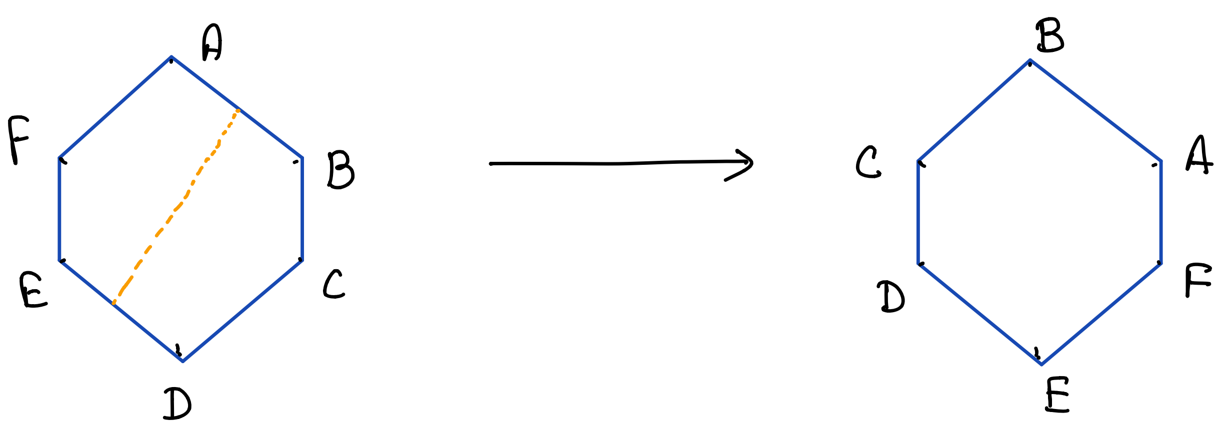


But we can do the same thing with any pair of opposite vertices. So we get

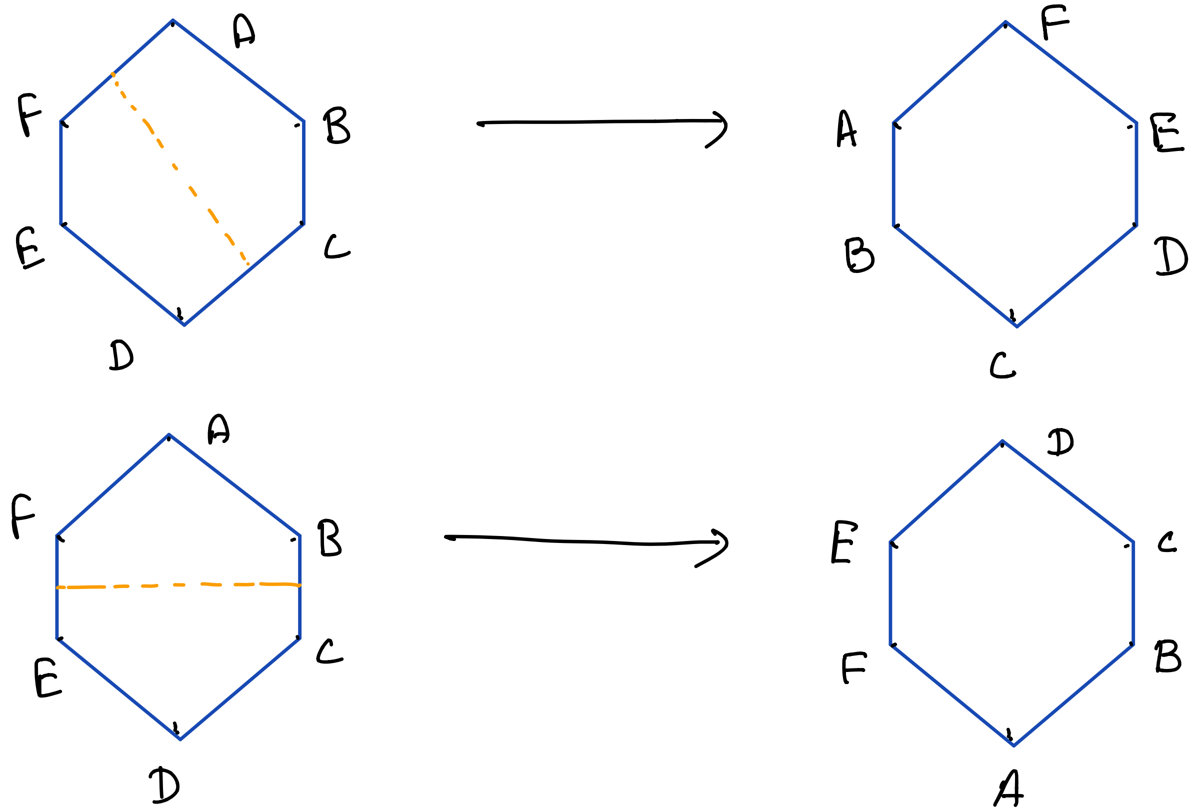




So we get 3 more elements in D_6 and so we have 9 elements so far. Now, we can draw an axis from the middle of edge AB to middle of edge ED and then flip about that



But we can do the same thing with any pair of opposite edges. So we get



So we got all the 12 elements in D_6.

The situation in D_n, n even is the same.

Hence D_n has $2n$ elements and we can explicitly describe them too.

Also note that we can call all the above flippings a symmetry because our polygon is regular.